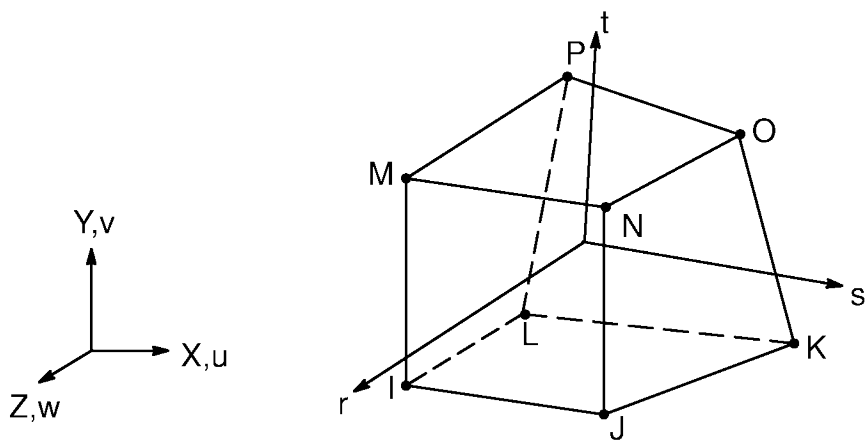
14.58 HYPER58 — 3–D 8–Node Mixed U–P Hyperelastic Solid



Matrix or Vector	Shape Functions	Integration Points
Stiffness Matrix	Equations (12.8.18–1), (12.8.18–2) and (12.8.18–3)	2 x 2 x 2
Mass Matrix	Same as stiffness matrix	2 x 2 x 2
Thermal Load Vector	Same as stiffness matrix	Same as stiffness matrix
Pressure Load Vector	Same as stiffness matrix, specialized to the face	2 x 2

Load Type	Distribution
Element Temperature	Trilinear thru element
Nodal Temperature	Trilinear thru element
Pressure	Bilinear across each face

References: Oden(123), Sussman(124)

14.58.1 Other Applicable Sections

The hyperelastic material model (Mooney–Rivlin) is described in Section 4.5. Section 13.1 describes integration point locations.

14.58.2 Mixed Hyperelastic Element Derivation

A mixed formulation is used that utilizes a modified strain energy density containing hydrostatic pressure as an explicit solution variable. Since it uses separate interpolations for the displacements and the hydrostatic pressure, it is referred to as the u–p (displacement–pressure) formulation. The essentials of the u–p formulation are summarized below. For details see references Oden and Kikuchi(123), Sussman and Bathe(124), and Zienkiewicz et al.(125).

14.58.3 Modified Strain Energy Density

The u–p formulation starts with a modified potential that explicitly includes the pressure variables:

$$W + Q = W - \frac{1}{2K}\,(p - \overline{p})^{2} \qquad (14.58\text{–}1)$$

where:
- Q = energy augmentation due to volume constraint condition
- K = bulk modulus
- p = pressure obtainable from W alone
- $\overline{p}$ = separately interpolated pressure (output stress item HPRES)

The original potential, W, for a Mooney–Rivlin material, which would be applicable for slightly incompressible rubber–like materials, is given by equation (4.5–8). Note that the last term of equation (4.5–8) provides the pressure p.

The displacements are discretized using standard isoparametric interpolations, whereas the pressure $\overline{p}$ is discretized by a polynomial expansion of the following form without any association with any nodes.

$$\overline{p} = p_1 + p_2 s + p_3 t + p_4 st + - - - \qquad (14.58\text{–}2)$$

where:
- s, t = element coordinates in natural space

14.58.4 Finite Element Matrices

The finite element matrices in terms of the incremental displacements and pressures are given by:

$$\begin{bmatrix} K^{uu} & K^{up} \\ K^{pu} & K^{pp} \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{p} \end{Bmatrix} = \begin{Bmatrix} F \\ 0 \end{Bmatrix} - \begin{Bmatrix} R^u \\ R^p \end{Bmatrix} \tag{14.58–3}$$

where: {F} = external nodal forces

$\{\dot{u}\}$, $\{\dot{p}\}$ = displacement and pressure increments respectively

$\{R^u\}$ and $\{R^p\}$ are the Newton–Raphson restoring force vectors (elsewhere referred to as $\{F^{nr}\}$):

$$R^u_i = \frac{\partial}{\partial \dot{u}_i}\left[\int_{vol}\left(W - \frac{1}{2K}(p - \overline{p})^2\right) d(vol)\right] = \int_{vol} \overline{S}_{kl}\frac{\partial E_{kl}}{\partial \dot{u}_i}\, d(vol) \tag{14.58–4}$$

$$R^p_i = \frac{\partial}{\partial \dot{p}_i}\left[\int_{vol}\left(W - \frac{1}{2K}(p - \overline{p})^2\right) d(vol)\right] = \int_{vol} \frac{1}{K}(p - \overline{p})\frac{\partial \overline{p}}{\partial \dot{p}_i} d(vol) \tag{14.58–5}$$

$$K^{uu}_{ij} = \frac{\partial R^u_i}{\partial u_j} = \text{displacement–only stiffness}$$

$$= \int_{vol} C^{uu}_{klrs}\frac{\partial E_{kl}}{\partial \dot{u}_i}\frac{\partial E_{rs}}{\partial \dot{u}_j} d(vol) + \int_{vol} \overline{S}_{kl}\frac{\partial^2 E_{kl}}{\partial \dot{u}_i \partial \dot{u}_j}\, d(vol) \tag{14.58–6}$$

$$K^{up}_{ij} = \frac{\partial R^u_i}{\partial \dot{p}_j} = \frac{\partial R^p_j}{\partial u_i} = K^{pu}_{ji} = \text{displacement–pressure coupled stiffness}$$

$$= \int_{vol} \frac{1}{K}\frac{\partial p}{\partial E_{kl}}\frac{\partial E_{kl}}{\partial \dot{u}_i}\frac{\partial \overline{p}}{\partial \dot{p}_j}\, d(vol) \tag{14.58–7}$$

$$K^{pp}_{ij} = \frac{\partial R^u_i}{\partial \dot{p}_j} = \text{pressure–only stiffness}$$

$$= \int_{vol} \frac{\partial \overline{p}}{\partial \dot{p}_i}\left(-\frac{1}{K}\right)\frac{\partial \overline{p}}{\partial \dot{p}_j}\, d(vol) \tag{14.58–8}$$

In the above,

$$\bar{S}_{kl} = S_{kl} - \frac{1}{K}(p - \bar{p})\frac{\partial p}{\partial E_{kl}} \qquad (14.58\text{–}9)$$

$$C^{uu}_{klrs} = \frac{\partial^2 W}{\partial E_{kl}\partial E_{rs}} - \frac{1}{K}\frac{\partial p}{\partial E_{kl}}\frac{\partial p}{\partial E_{rs}} - \frac{1}{K}(p - \bar{p})\frac{\partial^2 p}{\partial E_{kl}\partial E_{rs}} \qquad (14.58\text{–}10)$$

where: C^{uu}_{klrs} = augmented incremental moduli

The new augmented stress tensor $\bar{S}_{kl}$ has the property that the pressure corresponding to these new stresses $\bar{S}_{kl}$ when added with the pressure computed directly from the displacement configuration equals the separately interpolated pressure.

14.58.5 Incompressibility

The analysis of rubber–like materials poses computational difficulties in that these materials are almost incompressible. The fact that the volume changes very little while the material undergoes large strains often leads to displacement locking. In the u–p hyperelastic elements this difficulty is circumvented by enforcing the incompressibility constraint through a constraint equation. This constraint equation relates the separately interpolated pressure $(\bar{p})$ (output quantity HPRES) to the pressure (p) computed from the displacements and attempts to maintain the volume constraint in an average integrated sense over an element.

To be effective, there should be enough pressure DOFs $\bar{p}_i$, but the number of $\bar{p}_i$ DOFs in a model must be smaller than the number of unconstrained kinematic DOFs u_i (UX, UY, etc.) in order to allow deformation to occur at all. As a guideline, the number of unconstrained kinematic DOFs should be at least twice the number of pressure DOFs for 2–D problems, and at least three times the number of pressure DOFs for axisymmetric or 3–D problems.

14.58.6 Instabilities in the Material Constitutive Law

Instability may sometimes occur due to real buckling, or it may occur due to the mathematical procedure used in the formulation. For example, the application of a load in a single step that leads to a very large strain, say 100% or more, may cause instability. Furthermore, if there is a complex variation of the hydrostatic pressure, the number of pressure DOFs may not be adequate to describe the behavior. This may lead to a local volume change, associated with a decrease in total energy. In those cases, local mesh refinement or the use of higher order elements is recommended.

14.58.7 Existence of Multiple Solutions

For nonlinear problems, more than one stable solution may exist for a given set of boundary conditions. The case of a hollow hemisphere with zero prescribed loads is an example of such multiple solutions. Here the two equilibrium solutions are: the undeformed stress–free state and the inverted self–equilibrating state. Stable equilibrium solutions do not pose any difficulty; however, if the equilibrium becomes unstable at some point (e.g. incipient buckling) during the analysis, the solution procedure might collapse.